7.7

Rational Expressions - Solving Rational Equations

Objective: Solve rational equations by identifying and multiplying by the least common denominator.

When solving equations that are made up of rational expressions we will solve them using the same strategy we used to solve linear equations with fractions. When we solved problems like the next example, we cleared the fraction by multiplying by the least common denominator (LCD)

Example 1.

$$\begin{array}{rl}
\frac{2}{3}x - \frac{5}{6} = \frac{3}{4} & \text{Multiply each term by LCD, 12} \\[6pt]
\frac{2(12)}{3}x - \frac{5(12)}{6} = \frac{3(12)}{4} & \text{Reduce fractions} \\[6pt]
2(4)x - 5(2) = 3(3) & \text{Multiply} \\
8x - 10 = 9 & \text{Solve} \\
\underline{+10\ +10} & \text{Add 10 to both sides} \\
8x = 19 & \text{Divide both sides by 8} \\
\overline{8}\quad \overline{8} & \\
x = \frac{19}{8} & \text{Our Solution}
\end{array}$$

We will use the same process to solve rational equations, the only difference is our LCD will be more involved. We will also have to be aware of domain issues. If our

LCD equals zero, the solution is undefined. We will always check our solutions in the LCD as we may have to remove a solution from our solution set.

Example 2.

$$\frac{5x+5}{x+2}+3x=\frac{x^2}{x+2} \quad \text{Multiply each term by LCD, } (x+2)$$

$$\frac{(5x+5)(x+2)}{x+2}+3x(x+2)=\frac{x^2(x+2)}{x+2} \quad \text{Reduce fractions}$$

$$\begin{array}{rl}
5x+5+3x(x+2)=x^2 & \text{Distribute} \\
5x+5+3x^2+6x=x^2 & \text{Combine like terms} \\
3x^2+11x+5=x^2 & \text{Make equation equal zero} \\
\underline{-x^2 \qquad\qquad -x^2} & \text{Subtract } x^2 \text{ from both sides} \\
2x^2+11x+5=0 & \text{Factor} \\
(2x+1)(x+5)=0 & \text{Set each factor equal to zero} \\
2x+1=0 \text{ or } x+5=0 & \text{Solve each equation} \\
\underline{-1-1} \qquad \underline{-5-5} & \\
2x=-1 \text{ or } x=-5 & \\
\overline{2} \quad \overline{2} \qquad\qquad & \\
x=-\frac{1}{2} \text{ or } -5 & \text{Check solutions, LCD can't be zero} \\
-\frac{1}{2}+2=\frac{3}{2} \qquad -5+2=-3 & \text{Neither make LCD zero, both are solutions} \\
x=-\frac{1}{2} \text{ or } -5 & \text{Our Solution}
\end{array}$$

The LCD can be several factors in these problems. As the LCD gets more complex, it is important to remember the process we are using to solve is still the same.

Example 3.

$$\frac{x}{x+2}+\frac{1}{x+1}=\frac{5}{(x+1)(x+2)} \quad \text{Multiply terms by LCD, } (x+1)(x+2)$$

$$\frac{x(x+1)(x+2)}{x+2}+\frac{1(x+1)(x+2)}{x+1}=\frac{5(x+1)(x+2)}{(x+1)(x+2)} \quad \text{Reduce fractions}$$

$$
\begin{aligned}
x(x+1)+1(x+2)&=5 && \text{Distribute}\\
x^2+x+x+2&=5 && \text{Combine like terms}\\
x^2+2x+2&=5 && \text{Make equatino equal zero}\\
\underline{-5-5} && && \text{Subtract 6 from both sides}\\
x^2+2x-3&=0 && \text{Factor}\\
(x+3)(x-1)&=0 && \text{Set each factor equal to zero}\\
x+3=0 \ \text{ or } \ x-1&=0 && \text{Solve each equation}\\
\underline{-3-3} \qquad \underline{+1+1} && &&\\
x=-3 \ \text{ or } \ x&=1 && \text{Check solutions, LCD can}'t\text{ be zero}\\
(-3+1)(-3+2)=(-2)(-1)&=2 && \text{Check } -3 \text{ in } (x+1)(x+2)\text{, it works}\\
(1+1)(1+2)=(2)(3)&=6 && \text{Check 1 in } (x+1)(x+2)\text{, it works}\\
x=-3 \ \text{ or } \ 1 && && \text{Our Solution}
\end{aligned}
$$

In the previous example the denominators were factored for us. More often we will need to factor before finding the LCD

Example 4.

$$
\begin{aligned}
\frac{x}{x-1}-\frac{1}{x-2}=\underset{(x-1)(x-2)}{\frac{11}{x^2-3x+2}} && \text{Factor denominator}\\
\text{LCD}=(x-1)(x-2) && \text{Identify LCD}\\
\frac{x(x-1)(x-2)}{x-1}-\frac{1(x-1)(x-2)}{x-2}=\frac{11(x-1)(x-2)}{(x-1)(x-2)} && \text{Multiply each term by LCD, reduce}\\
x(x-2)-1(x-1)=11 && \text{Distribute}\\
x^2-2x-x+1=11 && \text{Combine like terms}\\
x^2-3x+1=11 && \text{Make equation equal zero}\\
\underline{-11-11} && \text{Subtract 11 from both sides}\\
x^2-3x-10=0 && \text{Factor}\\
(x-5)(x+2)=0 && \text{Set each factor equal to zero}\\
x-5=0 \ \text{ or } \ x+2=0 && \text{Solve each equation}\\
\underline{+5+5} \qquad \underline{-2-2} &&\\
x=5 \ \text{ or } \ x=-2 && \text{Check answers, LCD can}'t\text{ be 0}\\
(5-1)(5-2)=(4)(3)=12 && \text{Check 5 in } (x-1)(x-2)\text{, it works}\\
(-2-1)(-2-2)=(-3)(-4)=12 && \text{Check } -2 \text{ in } (x-1)(x-2)\text{, it works}\\
x=5 \ \text{ or } \ -2 && \text{Our Solution}
\end{aligned}
$$

World View Note: Maria Agnesi was the first women to publish a math textbook in 1748, it took her over 10 years to write! This textbook covered everything from arithmetic thorugh differential equations and was over 1,000 pages!

If we are subtracting a fraction in the problem, it may be easier to avoid a future sign error by first distributing the negative through the numerator.

Example 5.

$$\frac{x-2}{x-3}-\frac{x+2}{x+2}=\frac{5}{8} \quad \text{Distribute negative through numerator}$$

$$\frac{x-2}{x-3}+\frac{-x-2}{x+2}=\frac{5}{8} \quad \text{Identify LCD, } 8(x-3)(x+2)\text{, multiply each term}$$

$$\frac{(x-2)8(x-3)(x+2)}{x-3}+\frac{(-x-2)8(x-3)(x+2)}{x+2}=\frac{5\cdot 8(x-3)(x+2)}{8} \quad \text{Reduce}$$

$$\begin{array}{rl}
8(x-2)(x+2)+8(-x-2)(x-3)=5(x-3)(x+2) & \text{FOIL}\\
8(x^2-4)+8(-x^2+x+6)=5(x^2-x-6) & \text{Distribute}\\
8x^2-32-8x^2+8x+48=5x^2-5x-30 & \text{Combine like terms}\\
8x+16=5x^2-5x-30 & \text{Make equation equal zero}\\
\underline{-8x-16} \qquad \underline{-8x-16} & \text{Subtract } 8x \text{ and } 16\\
0=5x^2-13x-46 & \text{Factor}\\
0=(5x-23)(x+2) & \text{Set each factor equal to zero}\\
5x-23=0 \text{ or } x+2=0 & \text{Solve each equation}\\
\underline{+23+23} \qquad \underline{-2-2} & \\
5x=23 \text{ or } x=-2 & \\
\overline{5} \quad \overline{5} \qquad\qquad & \\
x=\dfrac{23}{5} \text{ or } -2 & \text{Check solutions, LCD can't be 0}\\
8\left(\dfrac{23}{5}-3\right)\left(\dfrac{23}{5}+2\right)=8\left(\dfrac{8}{5}\right)\left(\dfrac{33}{5}\right)=\dfrac{2112}{25} & \text{Check } \dfrac{23}{5} \text{ in } 8(x-3)(x+2)\text{, it works}\\
8(-2-3)(-2+2)=8(-5)(0)=0 & \text{Check } -2 \text{ in } 8(x-3)(x+2)\text{, can't be 0!}\\
x=\dfrac{23}{5} & \text{Our Solution}
\end{array}$$

In the previous example, one of the solutions we found made the LCD zero. When this happens we ignore this result and only use the results that make the rational expressions defined.

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7.7 Practice - Solving Rational Equations

Solve the following equations for the given variable:

1) $3x-\frac{1}{2}-\frac{1}{x}=0$

2) $x+1=\frac{4}{x+1}$

3) $x+\frac{20}{x-4}=\frac{5x}{x-4}-2$

4) $\frac{x^2+6}{x-1}+\frac{x-2}{x-1}=2x$

5) $x+\frac{6}{x-3}=\frac{2x}{x-3}$

6) $\frac{x-4}{x-1}=\frac{12}{3-x}+1$

7) $\frac{2x}{3x-4}=\frac{4x+5}{6x-1}-\frac{3}{3x-4}$

8) $\frac{6x+5}{2x^2-2x}-\frac{2}{1-x^2}=\frac{3x}{x^2-1}$

9) $\frac{3m}{2m-5}-\frac{7}{3m+1}=\frac{3}{2}$

10) $\frac{4x}{2x-6}-\frac{4}{5x-15}=\frac{1}{2}$

11) $\frac{4-x}{1-x}=\frac{12}{3-x}$

12) $\frac{7}{3-x}+\frac{1}{2}=\frac{3}{4-x}$

13) $\frac{7}{y-3}-\frac{1}{2}=\frac{y-2}{y-4}$

14) $\frac{2}{3-x}-\frac{6}{8-x}=1$

15) $\frac{1}{x+2}-\frac{1}{2-x}=\frac{3x+8}{x^2-4}$

16) $\frac{x+2}{3x-1}-\frac{1}{x}=\frac{3x-3}{3x^2-x}$

17) $\frac{x+1}{x-1}-\frac{x-1}{x+1}=\frac{5}{6}$

18) $\frac{x-1}{x-3}+\frac{x+2}{x+3}=\frac{3}{4}$

19) $\frac{3}{2x+1}+\frac{2x+1}{1-2x}=1-\frac{8x^2}{4x^2-1}$

20) $\frac{3x-5}{5x-5}+\frac{5x-1}{7x-7}-\frac{x-4}{1-x}=2$

21) $\frac{x-2}{x+3}-\frac{1}{x-2}=\frac{1}{x^2+x-6}$

22) $\frac{x-1}{x-2}+\frac{x+4}{2x+1}=\frac{1}{2x^2-3x-2}$

23) $\frac{3}{x+2}+\frac{x-1}{x+5}=\frac{5x+20}{6x+24}$

24) $\frac{x}{x+3}-\frac{4}{x-2}=\frac{-5x^2}{x^2+x-6}$

25) $\frac{x}{x-1}-\frac{2}{x+1}=\frac{4x^2}{x^2-1}$

26) $\frac{2x}{x+2}+\frac{2}{x-4}=\frac{3x}{x^2-2x-8}$

27) $\frac{2x}{x+1}-\frac{3}{x+5}=\frac{-8x^2}{x^2+6x+5}$

28) $\frac{x}{x+1}-\frac{3}{x+3}=\frac{-2x^2}{x^2+4x+3}$

29) $\frac{x-5}{x-9}+\frac{x+3}{x-3}=\frac{-4x^2}{x^2-12x+27}$

30) $\frac{x-3}{x+6}+\frac{x-2}{x-3}=\frac{x^2}{x^2+3x-18}$

31) $\frac{x-3}{x-6}+\frac{x+5}{x+3}=\frac{-2x^2}{x^2-3x-18}$

32) $\frac{x+3}{x-2}+\frac{x-2}{x+1}=\frac{9x^2}{x^2-x-2}$

33) $\frac{4x+1}{x+3}+\frac{5x-3}{x-1}=\frac{8x^2}{x^2+2x-3}$

34) $\frac{3x-1}{x+6}-\frac{2x-3}{x-3}=\frac{-3x^2}{x^2+3x-18}$

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7.7

Answers - Solving Rational Equations

1) $-\frac{1}{2}, \frac{2}{3}$

2) $-3, 1$

3) 3

4) $-1, 4$

5) 2

6) $\frac{1}{3}$

7) -1

8) $-\frac{1}{3}$

9) -5

10) $-\frac{7}{15}$

11) $-5, 0$

12) $5, 10$

13) $\frac{16}{3}, 5$

14) 2, 13

15) -8

16) 2

17) $-\frac{1}{5}, 5$

18) $-\frac{9}{5}, 1$

19) $\frac{3}{2}$

20) 10

21) 0, 5

22) $-2, \frac{5}{3}$

23) $4, 7$

24) -1

25) $\frac{2}{3}$

26) $\frac{1}{2}$

27) $\frac{3}{10}$

28) 1

29) $-\frac{2}{3}$

30) -1

31) $\frac{13}{4}$

32) 1

33) -10

34) $\frac{7}{4}$

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